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COMPUTING SCIENCE

Follow the Money

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The rich get richer and the poor get poorer. You've heard that before. It is a maxim so often repeated, and so often confirmed by experience, that it begins to sound like a law of nature, as familiar and irresistible as gravity. And indeed perhaps there *is* some physical or mathematical rule governing the distribution of wealth in the world. No such general principle is going explain the specifics of *who* gets rich and poor, but it might illuminate the overall statistics.

This idea goes back at least a century to the work of the Italian economist Vilfredo Pareto, who tried to show that the income distribution in all cultures and countries has the same mathematical form. In recent years the topic has been taken up with renewed enthusiasm by a small band of "econophysicists," who apply principles of statistical mechanics to questions in economic theory. The essence of their approach is to study an economy as if it were a many-body physical system such as a gas. Just as random collisions between gas molecules give rise to macroscopic properties such as temperature and pressure, random encounters between individuals in an economic system might determine large-scale phenomena such as the distribution of wealth.

Some of the computational models for exploring these issues are remarkably easy to build and run. It takes just a few minutes' effort and a few lines of code. On the other hand, it's also remarkably easy to make subtle mistakes of implementation, as I'll have occasion to mention below. And the big challenge is not building the models but interpreting the results—deciding which kinds of random encounters might represent events in a real economy.

The Price Is Right

The economy being simulated in these models is a rather special one, based on pure, free-market trading. The exchange of assets is all that ever happens here; there is no production of new wealth, and no consumption either. Leaving out so much of the real economy is an obvious weakness, but there is a compensating advantage: What remains is a closed system. In the model, wealth is a conserved quantity, like energy or momentum. Because the total amount of wealth never changes, one person can get richer only if another grows poorer.

I find it helpful to think of this miniature economy in terms of a yard sale, where all the participants put their goods out on the lawn Saturday morning, then stroll up and down the street making trades with their neighbors. At the end of the day, after all transactions are completed, an auditor reviews everyone's inventory and calculates their new net worth.

Many economic models assume that all transactions occur at precisely the right price. Indeed, prices are correct by definition in such models: Whatever price is agreed to by a willing seller and a willing buyer is the value assigned to an asset. Given such perfect pricing, nothing interesting could ever happen in the yard-sale economy. I might trade my old toaster for your broken VCR, but if we negotiate the terms of the deal correctly, my net worth will not change in the slightest, and neither will yours.

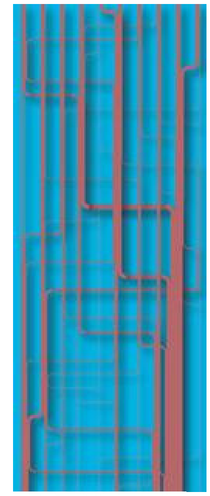
In practice, the assumption of perfect pricing seems a little unrealistic. Some buyers are more discerning than others, and some sellers are more persuasive. There are bargains to be had, and there are bad deals—concepts that could hardly exist if we did not agree that merchandise has a true and proper value, which does not always correspond exactly to the price paid.

Even slight departures from perfect pricing bring a new dynamic to the yard-sale model. If I buy your rusty wheelbarrow and pay more than it's worth, I am left slightly poorer after the transaction, and you are a little richer. Conversely, if I pay less than fair value, I gain a little, and you lose. In either case there has been a transfer of wealth, typically a small fraction of the price paid. These transfers are where the action is in the modeled economy; as a matter of fact, the model can ignore the transaction itself—there's no need to talk about toasters and wheelbarrows—and simply consider the net transfer of wealth.

The question is: What happens when this process is repeated many times? If some of the traders are shrewder than others, you would certainly expect them to do well in the long run; likewise the perennial suckers are going to lose their shirts. But suppose that everyone is equally skillful, so that who wins and who loses is purely a matter of chance. The amount of gain and loss is also determined at random—but it's always less than the total wealth of the poorer agent, so that traders never risk losing more than they own.

Before reading on, you might try to predict what will happen in such an economy. If everyone starts out with the same bankroll, how will the assets be distributed after many random exchanges? Will the levels of wealth remain uniform? Perhaps the system will evolve toward a Gaussian distribution, with most people having a middling amount of money, while a few are very poor and a few are rich?

Here is the answer given by the computer experiment: If trading continues long enough, essentially all the wealth winds up in the hands of one person. The yard-sale economy, as formulated in this model, is a winner-take-all lottery. The traders might just as well put all their goods in one big pile, and then roll the dice to decide who keeps it all. (Strictly speaking, one trader gets *all* the goods only if wealth is quantized—if there is some smallest unit of value below which one's worth falls to zero. If wealth can be subdivided indefinitely, the winner's share comes arbitrarily close to 100 percent but never quite gets there.)



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This condensation of all property in the hands of one individual is an economic catastrophe—something like the formation of a black hole in astrophysics. It's obviously bad news for the majority of the people, who are left penniless. But even if you happen to be the big winner, your victory may prove hollow. Although you have all the riches in the world, you can't buy a thing, because no one else has goods to sell. And you can't sell anything either, because no one has money to buy with. The whole economy is frozen.

Molecular Economics

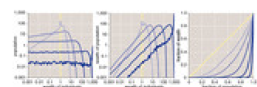
I first began experimenting with the yard-sale simulation after reading an article, "Wealth distributions in asset exchange models," by Slava Ispolatov, Paul L. Krapivsky and Sidney Redner of Boston University. The computer models described there seemed both intriguing and easy to re-create, and so I wrote a quick-and-dirty program to play with some of them. I was perplexed when my results were quite different from those reported in the article. A second look revealed that I had misread a key equation, so that my model differed from theirs in a small but crucial way. Later I found a paper by Anirban Chakraborti of the Saha Institute of Nuclear Physics in India that describes essentially the same model I had accidentally created.

At least two other groups of physicists have recently published work on related themes. In France, Jean-Philippe Bouchaud of the Centre d'etudes de Saclay and Marc Mézard of the Ecole normale supérieure have described "wealth condensation" in a somewhat different model. And Adrian Dragulescu and Victor M. Yakovenko of the University of Maryland have written on "the statistical mechanics of money."

A source of ideas for most of these models is the analogy between market economics and the kinetic theory of gases. The molecules of a gas are continually colliding with one another and exchanging energy, in much the way that randomly chosen buyers and sellers in an economic model exchange sums of money. Yet gases do not follow the evolutionary path of the yard-sale economy. An economic collapse, where one person sucks in all the money, corresponds to a gas where one molecule has all the kinetic energy, and the rest are standing still. Don't hold your breath waiting for that to happen.

Where the yard-sale model departs from the kinetic theory of gases is in the details of the exchange of wealth or energy. When two gas molecules collide, they can reapportion their energy in any way that leaves the total unchanged. If the molecules have energies a and b just before they collide, afterward they can have any combination of energies that add up to $a+b$. Translating this energy-redistribution process into financial terms yields a market in which the parties to a transaction combine their wealth and then randomly divide the total. A simulated economy based on this rule does not collapse the way the yard-sale model does; wealth remains spread throughout the population, although not uniformly so. The distribution follows an exponential curve: The number of people with wealth w is proportional to $e^{-w/T}$, where T is the temperature. (In the economic context, Dragulescu and Yakovenko identify the temperature with the average amount of money available to the participants.)

An exponential distribution crowds most of the people into the lower economic strata, but compared with the lopsided outcome of the yard-sale model, the degree of inequity is fairly mild. At least it's not an all-or-nothing economy. Furthermore, although the shape of the distribution is stable, individuals do not remain stationary within it: There are many rags-to-riches-to-rags stories in such a society. The gap between rich and poor seems less unfair if people have a reasonable chance of moving between these categories.



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An exponential distribution of wealth is clearly preferable to a winner-take-all outcome, and an economic model based on the kinetic theory of gases may also have a certain aesthetic appeal—at least to physicists. Nevertheless, the interpretation of the model is problematic. There is no obvious reason to expect economic agents to act like colliding molecules, and indeed the random repartitioning of kinetic energy is a fairly strange template for mercantile transactions. Applied in the yard-sale context, it suggests that when Bill Gates comes to browse among my lawn ornaments, he and I will pool all our assets and then randomly split up the pot.

One kind of financial transaction that might fit the pattern of the kinetic gas theory is marriage followed by divorce: This is a case where the parties do combine their holdings and later redivide them, although perhaps not quite randomly. In the corporate world, mergers and spin-offs might produce similar results.

Crime Doesn't Pay

The two models described so far lie at opposite poles along an axis defined by the amounts the trading parties put at risk. In the yard-sale model, the most that can be won or lost is the total wealth of the poorer partner. Since this model evolves toward a state where nearly everyone is impoverished, the typical transaction is extremely small. In the marriage-divorce model, in contrast, the entire fortunes of both partners are up for grabs.

Here is a recipe for a third model that occupies a middle ground. As in the yard-sale algorithm, pick two trading partners at random, and also randomly choose which of the partners is to lose (the donor) and which is to gain (the recipient). But instead of setting the size of the trade as a random fraction of the poorer player's wealth, make it a random fraction of the wealth of the donor. This rule still satisfies the commonsense constraint that you can never be made to pay more than you have. In each transaction you risk losing a random fraction of your own wealth, but you have a chance to gain a random fraction of the other person's fortune.

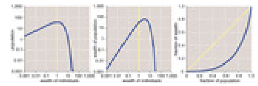
What kinds of real-world transactions might be described by this model? No doubt there are many plausible interpretations, but here is one that I find intriguing. A distinctive characteristic of the trading scheme is that the richer party always has more to lose and the poorer more to gain. Under these terms , any sensible person would try to do business only with wealthier partners, and no one would ever willingly choose to trade with a less-affluent person (assuming traders can gauge the wealth of their partners). Thus if trading between nonequals takes place at all, it must be by coercion or deception. In other words, what is being modeled here is theft and fraud.

When the theft-and-fraud model is allowed to run for many iterations, there is no economic collapse. The wealth distribution reaches an equilibrium on an exponential curve much like that seen in the marriage-and-divorce model. (I have no comment on this evidence that marriage and divorce have the same economic impact as larceny, nor will I speculate on why a world populated by bank robbers winds up with a fairer distribution of wealth than an economy of honest merchants.)

Beyond the Dreams of Avarice

The recent publications on asset-exchange models describe many more variations. Dragulescu and Yakovenko mention a family of models that differ among themselves only in the rule for choosing an amount of money to transfer. In one case it is a small fixed quantity; in another it is a random fraction of the trading pair's average wealth; in a third model the amount is a random fraction of the average wealth of the entire population. To avoid putting traders into debt or bankruptcy, Dragulescu and Yakovenko apply the meta-rule that if the loser cannot pay, the entire transaction is canceled. In all these models the equilibrium distribution has an exponential form, and there is no economic collapse.

Ispolatov, Krapivsky and Redner look at greedy or exploitative rules, where the wealthier party always wins the exchange (perhaps reflecting a situation where the poor have less bargaining power). When the amount transferred is a random fraction of the poorer's agent's wealth (as in the yard-sale model), the result is economic collapse, with all funds gravitating toward one person. Of course it's hardly a surprise that systematic greed yields a harsh outcome. The surprise is that this obviously biased rule is no worse than the symmetrical rule in the yard-sale model.



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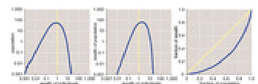
Chakraborti looks at the effect of savings, allowing traders to hold back some of their capital from the market. In the yard-sale economy, savings cannot forestall a collapse. Reserving a fixed sum of money shifts the minimum wealth up from zero but does not alter the dynamics of the model. Saving a fixed fraction of wealth slows the collapse, but the winner still takes all in the end.

Several authors mention the effects of taxes, welfare and other explicit means of redistributing income. Imposing a tax on wealth prevents the implosion of the yard-sale economy *(see Figure 4)*, but the effects of an income tax are not so clear. I experimented with income taxes by collecting a percentage of each transaction and redistributing the proceeds in equal shares to all traders. A low tax rate does not protect against collapse, but models with tax rates higher than about 15 percent do seem to survive indefinitely. If there is a sharp threshold between these regimes, I have been unable to identify it.

Trading with Zeno

The various economic models discussed here differ in many details, but they can be classified in two broad families: those where the economy falls into a black hole, with one trader acquiring nearly everything of value, and those where the distribution of wealth reaches some stable equilibrium. What is the root of the difference?

Dragulescu and Yakovenko point out that transactions like those in the yard-sale model break time-reversal symmetry. For an example of a transaction rule that *is* reversible, consider the marriage-and-divorce model. Lumping together two fortunes and then splitting the sum is a process that works the same both forward and backward. If two traders report that they have $5 and $3 at one moment, and $7 and $1 at another moment, with a single transaction between these states, you can't tell which report is earlier and which is later. The lumping-and-splitting rule could apply in either direction. In the yard-sale model, on the other hand, the crucial step is taking the minimum of the two amounts, and reversing this operation cannot always restore the initial configuration. A transaction carried out under the yard-sale rule can go from the $5-and-$3 state to the $7-and-$1 state, but not the other way.



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The irreversibility of the yard-sale rule acts as a kind of ratchet: Once the economy wanders into a state with an unbalanced distribution of wealth, it takes a long while to find its way out again. To see more clearly how the ratchet works, consider an even simpler model—an economy pared down to just two participants. Now the changing fortunes of either trader can be represented by a random walk along a line extending from zero to the total wealth available. All activity stops if the trader reaches either end of this line. A random walk that takes steps of uniform length is guaranteed to hit an end point sooner or later (a fate known as gambler's ruin). But this is *not* what is going on in the yard-sale model. There the steps are not of fixed size; because transactions are limited to the lesser of the trading partners' assets, the steps get smaller as the walk approaches either end point. If there is no smallest unit of currency, the random walk becomes a "Zeno walk," which spends most of its time in the neighborhood of an end point but never actually gets there.

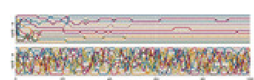
To simplify the model still further, we can take a Zeno walk on the interval from 0 to 1, choosing to go left or right at random but letting the step size always be half the distance to the nearer end point (rather than a random fraction of this distance). If we begin at the point 11/42, the initial step size is 11/44. Suppose the first move is to the right, reaching the point 31/44. Now the step size is 11/48. If we turn back to the left, we do not return to our starting point but instead stop at 51/48. Where will we wind up after *n* steps? The probability distribution for this process has an intricate fractal structure, so there is no simple answer, but the likeliest landing places get steadily closer to the end points of the interval as *n* increases. This skewed probability distribution is the ratchetlike mechanism that drives the yard-sale model to states of extreme imbalance.

Fair Trade

Models of the market economy may lead to some cute mathematics, but do they have the slightest connection with the price of peas in the real world? Can they predict the actual distribution of wealth observed in human societies?

As it happens, the shape of the actual distribution is uncertain and controversial. Most of the available data concern the distribution of income, which is not quite the same as the distribution of wealth. Pareto, 100 years ago, argued that the income distribution obeys a power law, so that the proportion of people whose income is at least x varies as x^{-a}; Pareto believed that the exponent a is a universal constant with a value of about 2.5. Other economists have proposed a log-normal income curve, meaning that the distribution of the logarithm of income is Gaussian.

The model of Bouchaud and Mézard (which includes investment earnings as well as trade) yields a Pareto-like power law for the wealth distribution. Some of the "greedy" models of Ispolatov, Krapivsky and Redner also appear to fit a power-law curve. But the models drawn most directly from the kinetic theory of gases predict an exponential distribution of wealth. Dragulescu and Yakovenko argue that the middle part of the actual wealth distribution is indeed exponential, with a "Pareto tail" in the highest tax brackets. All the computational models are so crude, however, and the empirical measurements are so uncertain, that curve-fitting inspires little confidence.



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Also unclear is whether events comparable to the collapse of the yard-sale model can happen in a real economy. Societies where a small elite controls almost all the property, while the rest of the people are destitute, are all too common. But does this situation result from a mathematical instability in the system of trade, or is there a simpler explanation, such as mere malice and greed? In any case, economic collapse seems never to go to completion in the real world, as it does in the models. Tycoons amass immense fortunes, but no ever one goes home with *all* the marbles. (Bill Gates holds much less than 1 percent of the world's wealth.)

Rather than trying to match the output of the models to economic statistics, it might be more fruitful to examine real-world economic practices for signs of the basic mechanisms that underlie the models. In particular, the fatal feature of the yard-sale model is the rule limiting the size of a transaction to the wealth of the lesser trading partner. The rule appears to be perfectly fair and symmetrical, and yet it has the effect that the farther you fall through the economic strata, the harder you'll find it to climb back up.

Is such a rule likely to be enforced in everyday commerce? Not always. It is clearly violated in many forms of gambling and speculation, where the whole point of the transaction is the hope of gaining more than you put at risk. Doubtless there are other exceptions as well. For the most part, though, those of us with less money are limited to smaller-scale buying and selling. And the lower the ceiling on your economic activity, the slower your progress up through the ranks. When I buy a new car, I have little chance—no matter how shrewdly I bargain—of significantly altering the balance of assets between me and General Motors.

Explaining the distribution of wealth among individuals is not the only possible application of the trading models. They might in fact be better suited to describing relations among companies, where a sudden consolidation of wealth could be interpreted as the emergence of a monopoly.

Beyond the corporate world, there is the question of whether the models might have anything to say about commerce among nations, and the ongoing debate over free markets, fair trade and a "level playing field." If some mechanism like that of the yard-sale model is truly at work, then markets might very well be free and fair, and the playing field perfectly level, and yet the outcome would almost surely be that the rich get richer and the poor get poorer. You've heard that before.